

HOMEOMORPHISM AND DIFFEOMORPHISM GROUPS OF NON-COMPACT MANIFOLDS WITH THE WHITNEY TOPOLOGY

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ABSTRACT. For a non-compact n -manifold M let $\mathcal{H}(M)$ be the group of homeomorphisms of M endowed with the Whitney topology and $\mathcal{H}_c(M)$ the subgroup of $\mathcal{H}(M)$ consisting of homeomorphisms with compact support. It is shown that the group $\mathcal{H}_c(M)$ is locally contractible and the identity component $\mathcal{H}_0(M)$ of $\mathcal{H}(M)$ is an open normal subgroup in $\mathcal{H}_c(M)$. This induces the topological factorization $\mathcal{H}_c(M) \approx \mathcal{H}_0(M) \times \mathcal{M}_c(M)$ for the mapping class group $\mathcal{M}_c(M) = \mathcal{H}_c(M)/\mathcal{H}_0(M)$ with the discrete topology. Furthermore, for any non-compact surface M , the pair $(\mathcal{H}(M), \mathcal{H}_c(M))$ is locally homeomorphic to $(\square^\omega l_2, \square^\omega l_2)$ at the identity id_M of M . Thus the group $\mathcal{H}_c(M)$ is an $(l_2 \times \mathbb{R}^\infty)$ -manifold. We also study topological properties of the group $\mathcal{D}(M)$ of diffeomorphisms of a non-compact smooth n -manifold M endowed with the Whitney C^∞ -topology and the subgroup $\mathcal{D}_c(M)$ of $\mathcal{D}(M)$ consisting of all diffeomorphisms with compact support. It is shown that the pair $(\mathcal{D}(M), \mathcal{D}_c(M))$ is locally homeomorphic to $(\square^\omega l_2, \square^\omega l_2)$ at the identity id_M of M . Hence the group $\mathcal{D}_c(M)$ is a topological $(l_2 \times \mathbb{R}^\infty)$ -manifold for any dimension n .

1. INTRODUCTION

In this paper we study topological properties of groups of homeomorphisms and diffeomorphisms of non-compact manifolds endowed with the Whitney topology and recognize their local topological type.

For a σ -compact n -manifold M possibly with boundary let $\mathcal{H}(M)$ denote the group of homeomorphisms of M endowed with the Whitney topology and $\mathcal{H}_c(M)$ the subgroup of $\mathcal{H}(M)$ consisting of homeomorphisms with compact support. In this topology, each $f \in \mathcal{H}(M)$ has the fundamental neighborhood system

$$\mathcal{U}(f) = \{g \in \mathcal{H}(M) : (f, g) \prec \mathcal{U}\} \quad (\mathcal{U} \in \text{cov}(M)),$$

where $\text{cov}(M)$ is the set of open covers of M and the notation $(f, g) \prec \mathcal{U}$ means that f and g are \mathcal{U} -near (i.e., every point $x \in M$ admits a set $U \in \mathcal{U}$ with $f(x), g(x) \in U$) (Proposition 3.1). The *support* of $h \in \mathcal{H}(M)$ is defined by

$$\text{supp}(h) = \text{cl}\{x \in M : h(x) \neq x\}.$$

The group $\mathcal{H}(M)$ is a topological group and the identity component $\mathcal{H}_0(M)$ of $\mathcal{H}(M)$ lies in the subgroup $\mathcal{H}_c(M)$ (Proposition 3.3(1)).

In case M is a compact n -manifold, the Whitney topology on the group $\mathcal{H}(M)$ coincides with the compact-open topology. Hence, $\mathcal{H}(M)$ is completely metrizable and locally contractible ([10], [11]). When M is a compact surface (or a finite graph), the group $\mathcal{H}(M)$ is an l_2 -manifold (the case of finite graphs is the result of [2]; the case of compact surfaces is a combination of the results of [23], [14] and [30]). For $n \geq 3$, it is still an open problem if the homeomorphism group $\mathcal{H}(M)$ of a compact n -manifold M is an l_2 -manifold [3], which is called the Homeomorphism Group Problem [33].

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In this paper we are concerned with the case that M is a noncompact σ -compact n -manifold. In this case the Whitney topology on $\mathcal{H}(M)$ is still very important in Geometric Topology, but it has rather bad local properties. Our observations in this paper mean that the subgroup $\mathcal{H}_c(M)$ of $\mathcal{H}(M)$ is quite nice from the topological viewpoint. First we recall basic facts on the local topological models of the groups $\mathcal{H}(M)$ and $\mathcal{H}_c(M)$.

A *Fréchet space* is a completely metrizable locally convex topological linear space and an LF-space is the direct (or inductive) limit of increasing sequence of Fréchet spaces in the category of locally convex topological linear spaces. A topological characterization of LF-spaces is given in [7]. The simplest non-trivial example of an LF-space is \mathbb{R}^∞ , the direct limit of the tower

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \dots,$$

where each space \mathbb{R}^n is identified with the hyperplane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. In [24] P. Mankiewicz studied LF-spaces and proved that an infinite-dimensional separable LF-space is homeomorphic to (\approx) either \mathbb{R}^∞ , l_2 or $l_2 \times \mathbb{R}^\infty$. The space $l_2 \times \mathbb{R}^\infty$ is homeomorphic to the countable small box power $\square^\omega l_2$ of the Hilbert space l_2 , which is a subspace of the box power $\square^\omega l_2$. One should notice that the box power $\square^\omega l_2$ is neither locally connected, nor sequential, nor normal (see [17], [34]).

In the papers [4] and [5] we have already shown that

$$(\mathcal{H}(\mathbb{R}), \mathcal{H}_c(\mathbb{R})) \approx (\mathcal{H}([0, \infty)), \mathcal{H}_c([0, \infty))) \approx (\square^\omega l_2, \square^\omega l_2).$$

Moreover, it is proved in [5] that if M is a non-compact separable graph then $\mathcal{H}_c(M)$ is an $(l_2 \times \mathbb{R}^\infty)$ -manifold. Thus, as a non-compact version of the Homeomorphism Group Problem, we can expect the following:

Conjecture. *For any non-compact σ -compact n -manifold M possibly with boundary, the pair $(\mathcal{H}(M), \mathcal{H}_c(M))$ is locally homeomorphic to $(\square^\omega l_2, \square^\omega l_2)$ at the identity id_M of M . In particular, the group $\mathcal{H}_c(M)$ is an $(l_2 \times \mathbb{R}^\infty)$ -manifold.*¹

Here, we say that a pair (X', X) of topological spaces $X \subset X'$ is *locally homeomorphic* to a pair (Y', Y) if each point $x \in X$ has an open neighborhood $U \subset X'$ such that the pair $(U, U \cap X)$ is homeomorphic to the pair $(V, V \cap Y)$ for some open set $V \subset Y'$.

In this paper we first show that the group $\mathcal{H}_c(M)$ is locally contractible in any dimension n .

Proposition 1. *For any σ -compact n -manifold M possibly with boundary, the group $\mathcal{H}_c(M)$ is locally contractible.*

Thus the identity component $\mathcal{H}_0(M)$ is an open normal subgroup of $\mathcal{H}_c(M)$ and it induces the topological factorization

$$\mathcal{H}_c(M) \approx \mathcal{H}_0(M) \times \mathcal{M}_c(M),$$

where $\mathcal{M}_c(M) = \mathcal{H}_c(M)/\mathcal{H}_0(M)$ is the *mapping class group* of M with the discrete topology.

As mentioned above, the conjecture above has been proved in the case $n = 1$, see [5]. Here we solve the conjecture affirmatively in the case $n = 2$.

Theorem 2. *Suppose M is a non-compact σ -compact 2-manifold possibly with boundary. Then the pair $(\mathcal{H}(M), \mathcal{H}_c(M))$ is locally homeomorphic to $(\square^\omega l_2, \square^\omega l_2)$ at the identity id_M of M . In particular, the group $\mathcal{H}_c(M)$ is an $(l_2 \times \mathbb{R}^\infty)$ -manifold.*

¹Because of the **non-paracompactness** of $\square^\omega l_2$, we avoid to say “ $\square^\omega l_2$ -manifold” or “ $(\square^\omega l_2, \square^\omega l_2)$ -manifold pair”.

Any σ -compact 2-manifold M possibly with boundary admits a PL-structure [26]. When M is equipped with a PL-structure, the symbol $\mathcal{H}^{PL}(M)$ denotes the subgroup of $\mathcal{H}(M)$ consisting of PL-homeomorphisms with respect to this PL-structure. A subspace A of a space X is said to be *homotopy dense* (abbrev. HD) if there exists a homotopy $\phi_t : X \rightarrow X$ such that $\phi_0 = \text{id}_X$ and $\phi_t(X) \subset A$ ($t \in (0, 1]$).

Proposition 3. *Suppose M is a non-compact σ -compact PL 2-manifold possibly with boundary. Then the subgroup $\mathcal{H}_c^{PL}(M)$ is homotopy dense in $\mathcal{H}_c(M)$.*

We also study the local topological type of groups of diffeomorphisms of non-compact smooth manifolds endowed with the Whitney C^∞ -topology. For a smooth n -manifold M , let $\mathcal{D}(M)$ denote the group of all diffeomorphisms of M endowed with the Whitney C^∞ -topology. Let $\mathcal{D}_0(M)$ denote the identity component of the diffeomorphism group $\mathcal{D}(M)$ and $\mathcal{D}_c(M)$ denote the subgroup of $\mathcal{D}(M)$ consisting of all diffeomorphisms of M with compact support.

Theorem 4. *For a non-compact σ -compact smooth n -manifold M without boundary, the pair $(\mathcal{D}(M), \mathcal{D}_c(M))$ is locally homeomorphic to $(\square^\omega l_2, \square^\omega l_2)$ at the identity id_M of M . In particular, the group $\mathcal{D}_c(M)$ is a topological $(l_2 \times \mathbb{R}^\infty)$ -manifold.*

This implies that the identity component $\mathcal{D}_0(M)$ is an open normal subgroup of $\mathcal{D}_c(M)$ and it induces the topological factorization

$$\mathcal{D}_c(M) \approx \mathcal{D}_0(M) \times \mathcal{M}_c^\infty(M), \quad \mathcal{M}_c^\infty(M) = \mathcal{D}_c(M)/\mathcal{D}_0(M).$$

In [8] we have shown that $(\mathcal{D}(\mathbb{R}), \mathcal{D}_c(\mathbb{R})) \approx (\square^\omega l_2, \square^\omega l_2)$. In the succeeding paper [6], we determine the global topological types of the groups $\mathcal{H}_c(M)$ for non-compact surfaces M and the groups $\mathcal{D}_c(M)$ for some kind of non-compact smooth n -manifolds M .

This paper is organized as follows: Section 2 contains the basic facts on the box products and the small box products, and Section 3 contains generalities on homeomorphism groups with the Whitney topology. In Section 4 we introduce some fundamental notations on transformation groups and in Section 5 we formulate the notion of strong topology on transformation groups and study some basic properties. In Section 6 we apply these results to groups of homeomorphisms and diffeomorphisms of noncompact manifolds and prove Theorems 2 and 4 together with Propositions 1 and 3.

2. BOX AND SMALL BOX PRODUCTS

In this section we recall some basic properties on box products and small box products. Let ω and \mathbb{N} denote the sets of non-negative integers and positive integers, respectively.

Definition 2.1. (1) The *box product* $\square_{n \in \omega} X_n$ of a sequence of topological spaces $(X_n)_{n \in \omega}$ is the countable product $\prod_{n \in \omega} X_n$ endowed with the box topology generated by the base consisting of boxes $\prod_{n \in \omega} U_n$, where U_n is an open subset of X_n .

(2) The *small box product* $\square_{n \in \omega} X_n$ of a sequence of pointed spaces $(X_n, *_n)_{n \in \omega}$ is the subspace of $\square_{n \in \omega} X_n$ defined by

$$\square_{n \in \omega} X_n = \{(x_n)_{n \in \omega} \in \square_{n \in \omega} X_n : \exists m \in \omega \forall n \geq m, x_n = *_n\}.$$

(3) The pair $(\square_{n \in \omega} X_n, \square_{n \in \omega} X_n)$ is denoted by the symbol $(\square, \square)_{n \in \omega} X_n$.

The small box product $\square_{n \in \omega} X_n$ has a canonical distinguished point $(*_n)_{n \in \omega}$. For a sequence of subsets $A_n \subset X_n$ ($n \in \omega$), let

$$\square_{n \in \omega} A_n = \square_{n \in \omega} X_n \cap \square_{n \in \omega} A_n,$$

where it is not assumed that $*_n \in A_n$. If $*_n \notin A_n$ for infinitely many $n \in \omega$ then $\square_{n \in \omega} A_n = \emptyset$. Identifying $\prod_{i \leq n} X_i$ with the closed subspace $\{(x_i)_{i \in \omega} \in \square_{i \in \omega} X_i : x_i = *_i \text{ (} i > n \text{)}\}$, we can regard $\square_{n \in \omega} X_n = \bigcup_{n \in \omega} \prod_{i \leq n} X_i$. When $X_n = X$ for all $n \in \omega$, we write $\square^\omega X$ and $\square^\omega X$ instead of $\square_{n \in \omega} X_n$ and $\square_{n \in \omega} X_n$, which are called the *box power* and the *small box power* of X , respectively. Then we can regard $\square^\omega X = \bigcup_{n \in \omega} X^n$, where $X \subset X^2 \subset X^3 \subset \dots$.

Proposition 2.2. *If each finite product $\prod_{i \leq n} X_i$ is paracompact, then the small box product $\square_{i \in \omega} X_i$ is also paracompact.*

Proof. By the characterization of paracompactness (Theorem 5.1.11 in [12]), it suffices to show that every open cover \mathcal{U} of $\square_{i \in \omega} X_i$ has a σ -locally finite open refinement. For each $n \in \mathbb{N}$, we shall construct a locally finite open collection \mathcal{V}_n in $\square_{i \in \omega} X_i$ which covers $\prod_{i \leq n} X_i$ and refines \mathcal{U} . Each $x \in \prod_{i \leq n} X_i$ has a basic open neighborhood $\square_{i \in \omega} U_i^x$ which is contained some member of \mathcal{U} . Then $\{\prod_{i \leq n} U_i^x : x \in \prod_{i \leq n} X_i\}$ is an open cover of $\prod_{i \leq n} X_i$, which has a locally finite open refinement \mathcal{U}_n . For each $U \in \mathcal{U}_n$, choose $x \in \prod_{i \leq n} X_i$ so that $U \subset \prod_{i \leq n} U_i^x$ and define $V_U = U \times \square_{i > n} U_i^x$. Then $\mathcal{V}_n = \{V_U : U \in \mathcal{U}_n\}$ ($n \in \omega$) satisfy the required conditions. Consequently, $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a σ -locally finite open refinement of \mathcal{U} . \square

A sequence of maps $\phi^n : (X_n, *_n) \rightarrow (Y_n, *_n)$ ($n \in \omega$) induces a continuous map

$$\square_{n \in \omega} \phi^n : \square_{n \in \omega} X_n \rightarrow \square_{n \in \omega} Y_n, \quad (\square_{n \in \omega} \phi^n)((x_n)_{n \in \omega}) = (\phi^n(x_n))_{n \in \omega}.$$

Lemma 2.3. *For a compact space K and a sequence of maps $\phi^n : (X_n \times K, \{*_n\} \times K) \rightarrow (Y_n, *_n)$ ($n \in \omega$), the map*

$$\Phi : (\square_{n \in \omega} X_n) \times K \rightarrow \square_{n \in \omega} Y_n, \quad \Phi((x_n)_{n \in \omega}, y) = (\phi^n(x_n, y))_{n \in \omega}.$$

is continuous.

Proof. The proof is straightforward. Take any point $((x_n)_{n \in \omega}, y)$ of $\square_{n \in \omega} X_n \times K$ and any open neighborhood V of $\Phi((x_n)_{n \in \omega}, y)$ in $\square_{n \in \omega} Y_n$. We may assume that V is of the form $V = \square_{n \in \omega} V_n$, where V_n is an open neighborhood of $\phi^n(x_n, y)$ in X_n . There exists $m \in \omega$ such that $x_n = *_n$ for $n > m$. For $n = 0, 1, \dots, m$, choose open neighborhoods U_n of x_n in X_n and W_n of y in K such that $\phi_n(U_n \times W_n) \subset V_n$. For $n > m$, since $\phi_n(\{*_n\} \times K) = \{*_n\} \subset V_n$ and K is compact, there exists an open neighborhood U_n of $x_n = *_n$ in X_n such that $\phi^n(U_n \times K) \subset V_n$. Then $U = \square_{n \in \omega} U_n$ and $W = \bigcap_{n=0}^m W_n$ are open neighborhoods of $(x_n)_n$ in $\square_{n \in \omega} X_n$ and y in K , respectively. Now, it is easy to see that $\Phi(U \times W) \subset V$. This completes the proof. \square

For example, a sequence of homotopies $\phi_t^n : (X_n, *_n) \rightarrow (Y_n, *_n)$ ($n \in \omega$) induces a homotopy

$$\square_{n \in \mathbb{N}} \phi_t^n : \square_{n \in \mathbb{N}} X_n \rightarrow \square_{n \in \mathbb{N}} Y_n.$$

This simple observation leads to some useful consequences.

Definition 2.4. A subspace A of a space X is called *homotopy dense* (abbrev. HD) in X rel. a subset A_0 of A if there exists a homotopy $\phi_t : X \rightarrow X$ ($t \in [0, 1]$) such that $\phi_0 = \text{id}_X$, $\phi_t|_{A_0} = \text{id}$ and $\phi_t(X) \subset A$ ($t \in (0, 1]$).

Proposition 2.5. *Let $(X_n, A_n, *_n)_{n \in \omega}$ be a sequence of pointed pair of spaces. If each A_n is HD in X_n rel. the point $*_n$, then $\square_{n \in \omega} A_n$ is HD in $\square_{n \in \omega} X_n$.*

Remark 2.6. (1) Suppose X is a metrizable space, $A_0 \subset A \subset X$ and A_0 is a closed subset of X . If A is HD in X , then A is HD in X rel. A_0 .

(2) Suppose G is a topological group and H is a subgroup of G . If H is HD in G , then H is HD in G rel. the identity element e of G .

Definition 2.7. (1) A subspace A of a space X is called *contractible* in X (rel. a point $a \in A$) if there exists a homotopy $\phi_t : A \rightarrow X$ ($t \in [0, 1]$) such that $\phi_0 = \text{id}_A$ and $\phi_1(A)$ is a singleton ($\phi_t(a) = a$ for every $t \in [0, 1]$, whence $\phi_1(A) = \{a\}$).

(2) A space X is called (*strongly*) *locally contractible* at $x \in X$ if every neighborhood U of x contains a neighborhood V of x which is contractible in U (rel. x).

(3) A pointed space $(X, *)$ is said to be (i) *locally contractible* if X is locally contractible at any point of X and strongly locally contractible at the point $*$, and (ii) *contractible* if X is contractible in X rel. $*$.

Proposition 2.8. *If pointed spaces $(X_n, *_n)$ ($n \in \omega$) are (locally) contractible, then the small box product $\square_{n \in \omega} X_n$ is also (locally) contractible as a pointed space.*

Remark 2.9. A space X is called *semi-locally contractible* at a point $x \in X$ if x has a neighborhood V in X which contracts in X . It is easy to see that if a topological group G is semi-locally contractible at the identity element $e \in G$ then G is strongly locally contractible at every $x \in G$, hence the pointed space (G, e) is locally contractible. Indeed, if $h : V_0 \times [0, 1] \rightarrow G$ is a contraction of a neighborhood V_0 of $e \in G$, then we can define a contraction $h' : V_0 \times [0, 1] \rightarrow G$ by $h'(x, t) = h(e, t)^{-1} h(x, t)$. Since $h'(\{e\} \times [0, 1]) = \{e\}$, every neighborhood U of e contains a neighborhood V of e such that $h'(V \times [0, 1]) \subset U$. Then the restriction $h'|_{V \times [0, 1]}$ is a contraction of V in U fixing the identity element e . Since the topological group G is homogeneous, it follows that G is strongly locally contractible at every $x \in G$.

Finally we discuss the box products of topological groups. As usual, we regard a topological group as a pointed space by distinguishing the identity element. For topological groups $(G_n)_{n \in \omega}$, the box product $\square_{n \in \omega} G_n$ is a topological group under the coordinatewise multiplication, and the small box product $\square_{n \in \omega} G_n$ is a topological subgroup of $\square_{n \in \omega} G_n$.

Suppose G is a topological group with the identity element $e \in G$. Any sequence $(G_n)_{n \in \omega}$ of subgroups of G induces the natural multiplication map

$$p : \square_{n \in \omega} G_n \rightarrow G, \quad p(x_0, \dots, x_k, e, e, \dots) = x_0 \cdot x_1 \cdots x_k.$$

Lemma 2.10. *The map p is continuous.*

Proof. Fix any point $x = (x_0, \dots, x_k) \in \square_{n \in \omega} G_n$ and take any neighborhood V of its image $p(x) = x_0 \cdots x_k$ in G . Replacing x by a longer sequence if necessary, we can assume that $x_k = e$. By the continuity of the group multiplication, find a sequence of neighborhoods U_n of x_n in G , $n \leq k$, such that

$$U_0 \cdot U_1 \cdots U_{k-1} \cdot U_k \cdot U_k \subset V.$$

Now, inductively construct a decreasing sequence $(U_n)_{n > k}$ of open neighborhoods of e in G such that $U_n \cdot U_n \subset U_{n-1}$ for all $n > k$. Such a choice will guarantee that

$$U_0 \cdots U_{k-1} \cdot U_k \cdot U_{k+1} \cdots U_n \subset U_0 \cdots U_{k-1} \cdot U_k \cdot U_k \subset V \quad (n > k).$$

Then the set $U = \square_{n \in \omega} U_n \cap \square_{n \in \omega} G_n$ is an open neighborhood of x in $\square_{n \in \omega} G_n$ and

$$p(U) \subset \bigcup_{n > k} (U_0 \cdots U_n) \subset V,$$

which proves the continuity of p at x . \square

Let $p : X \rightarrow Y$ be a continuous map. A *local section* of p at $y \in Y$ is a map $s : V \rightarrow X$ defined on a neighborhood V of y in Y such that $ps = \text{id}$. When $V = Y$, the map s is called a *section* of p .

Lemma 2.11. *Suppose $(G_n)_{n \in \omega}$ is a sequence of subgroups of G such that $G_n \subset G_{n+1}$ for each $n \in \omega$ and $G = \bigcup_{n \in \omega} G_n$. If the multiplication map $p : \square_{n \in \omega} G_n \rightarrow G$ has a local section at e , then the following hold:*

- (1) *The map p has a local section at any point of G and a local section s at e with $s(e) = (e, e, \dots)$.*
- (2) *If each G_n is locally contractible, then so is G .*
- (3) *Suppose G is paracompact and H is a subgroup of G . If $H_n = H \cap G_n$ is HD in G_n for each $n \in \omega$, then H is HD in G .*

Proof. (1) The verification is simple and omitted.

(2) By Remark 2.9 and Proposition 2.8, the small box product $\square_{n \in \omega} G_n$ is locally contractible. Since the map p has a local section at any point, the group G is also locally contractible.

(3) Since G is paracompact, it suffices to show that each $g \in G$ has an open neighborhood U in G with a homotopy $\phi_t : U \rightarrow G$ such that $\phi_0 = \text{id}$ and $\phi_t(U) \subset H$ ($t \in (0, 1]$). By (1) the map p admits a local section $s : U \rightarrow \square_{n \in \omega} G_n$ at the point g . By Remark 2.6 (2) and Proposition 2.5, the small box product $\square_{n \in \omega} H_n$ is HD in $\square_{n \in \omega} G_n$ by an absorbing homotopy ψ_t . Then the homotopy ϕ_t is defined by $\phi_t = p\psi_t s$. \square

3. BASIC PROPERTIES OF HOMEOMORPHISM GROUPS WITH THE WHITNEY TOPOLOGY

In this section, we list some basic properties of the Whitney topology on homeomorphism groups. For any topological space M , let $\mathcal{H}(M)$ denote the group of homeomorphisms of M endowed with the Whitney topology. This topology is generated by the subsets

$$\mathcal{U}(h) = \{g \in \mathcal{H}(M) : (h, g) \prec \mathcal{U}\}, \quad (h \in \mathcal{H}(M), \mathcal{U} \in \text{cov}(M)),$$

and each $h \in \mathcal{H}(M)$ has the neighborhood basis $\mathcal{U}(h)$ ($\mathcal{U} \in \text{cov}(M)$). On the space $C(X, Y)$ of all continuous functions from X to Y , the Whitney topology is usually defined as the *graph topology* or the WO^0 -topology, that is, it is generated by

$$\Gamma_U = \{f \in C(X, Y) : \Gamma_f \subset U\},$$

where U runs through all open sets in $X \times Y$ and $\Gamma_f = \{(x, f(x)) : x \in X\}$ is the graph of $f \in C(X, Y)$ (e.g., see [20, §41]). The *graph topology* or the WO^0 -topology on $\mathcal{H}(M)$ is the subspace topology inherited from the space $C(M, M)$ with this topology. In the space $C(X, Y)$, the graph topology is not generated by the sets

$$\mathcal{U}(f) = \{g \in C(X, Y) : (f, g) \prec \mathcal{U}\} \quad (f \in C(X, Y), \mathcal{U} \in \text{cov}(Y)).$$

For completeness, we give a proof of the following:

Proposition 3.1. *For any topological space M , $\mathcal{U}(h)$ ($\mathcal{U} \in \text{cov}(M)$) is a neighborhood basis of $h \in \mathcal{H}(M)$ in the graph topology.*

Proof. Fix $h \in \mathcal{H}(M)$. (1) Let $W \subset M^2$ be an open set such that $\Gamma_h \subset W$. Each $x \in M$ has an open neighborhood U_x in M such that $U_x \times h(U_x) \subset W$. Since h is a homeomorphism, $\mathcal{U} = \{h(U_x) : x \in M\} \in \text{cov}(M)$. To see $\mathcal{U}(h) \subset \Gamma_W$, take any homeomorphism $g \in \mathcal{U}(h)$. For every point $z \in M$, there exists $y \in M$ such that $\{h(z), g(z)\} \subset h(U_y)$. Since h is a bijection, we have $z \in U_y$, hence $(z, g(z)) \in U_y \times h(U_y) \subset W$. This means that $\Gamma_g \subset W$ and hence $\mathcal{U}(h) \subset \Gamma_W$.

(2) Take any cover $\mathcal{U} \in \text{cov}(M)$. For each $x \in M$, choose $U_x \in \mathcal{U}$ with $h(x) \in U_x$. Then

$$W = \bigcup_{x \in M} h^{-1}(U_x) \times U_x \subset M \times M$$

is an open neighborhood of Γ_h in $M \times M$. To see $\Gamma_W \subset \mathcal{U}(h)$, take any $g \in \Gamma_W$. Since $\Gamma_g \subset W$, for any $y \in M$ we can find $x \in M$ such that $(y, g(y)) \in h^{-1}(U_x) \times U_x$ and hence $\{h(y), g(y)\} \subset U_x$. This means that $g \in \mathcal{U}(h)$. \square

For subspaces $K \subset L \subset M$, let $\mathcal{E}_K(L, M)$ denote the space of embeddings $f : L \rightarrow M$ with $f|_K = \text{id}_K$ endowed with the compact-open topology. In comparison with the Whitney topology (when M is Hausdorff), each $f \in \mathcal{E}_K(L, M)$ admits the fundamental neighborhood system:

$$\mathcal{U}(f, C) = \{g \in \mathcal{E}_K(L, M) : (f|_C, g|_C) \prec \mathcal{U}\} \quad (C \text{ is a compact subset of } L, \mathcal{U} \in \text{cov}(M)).$$

The group $\mathcal{H}(M)$ acts on $\mathcal{E}(L, M)$ by the left composition. When M is paracompact, every open cover of M admits a star-refinement. This remark leads to the following basic fact.

Proposition 3.2. *If M is paracompact,² then (i) $\mathcal{H}(M)$ is a topological group and (ii) the natural action of $\mathcal{H}(M)$ on $\mathcal{E}(L, M)$ is continuous.*

Proof. For the sake of completeness we include the proof.

(i) It follows from the definition of the Whitney topology on $\mathcal{H}(M)$ that the inversion $f \mapsto f^{-1}$ is continuous. So, it remains to check that the composition is continuous with respect to the Whitney topology. Given $f, g \in \mathcal{H}(M)$ and $\mathcal{U} \in \text{cov}(M)$, we should find $\mathcal{V}, \mathcal{W} \in \text{cov}(M)$ such that $f'g' \in \mathcal{U}(fg)$ for every $f' \in \mathcal{V}(f)$ and $g' \in \mathcal{W}(g)$, that is, $(f, f') \prec \mathcal{V}$ and $(g, g') \prec \mathcal{W}$ imply $(fg, f'g') \prec \mathcal{U}$. By the paracompactness of M , there is a cover $\mathcal{V} \in \text{cov}(M)$ with $St(\mathcal{V}) \prec \mathcal{U}$. Let $\mathcal{W} = f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$ and assume $(f, f') \prec \mathcal{V}$ and $(g, g') \prec \mathcal{W}$. Since $(fg, fg') \prec f(\mathcal{W}) = \mathcal{V}$ and $(fg', f'g') \prec \mathcal{V}$, it follows that $(fg, f'g') \prec St(\mathcal{V}) \prec \mathcal{U}$.

The assertion (ii) can be seen by the same argument. \square

The next proposition is the main result in this section. For a subset $L \subset M$, let $\mathcal{H}(M, L) = \{h \in \mathcal{H}(M) : h|_L = \text{id}\}$.

Proposition 3.3. *If M is paracompact then (1) $\mathcal{H}_0(M) \subset \mathcal{H}_c(M)$ and (2) every compact subspace $K \subset \mathcal{H}_c(M)$ is contained in $\mathcal{H}(M, M \setminus K)$ for some compact subset $K \subset M$.*

Proof. (1) It suffices to show that each $f \in \mathcal{H}(M) \setminus \mathcal{H}_c(M)$ can be separated from id_M by a clopen subset \mathcal{U} of $\mathcal{H}(M)$. For this purpose, we compare the space $\mathcal{H}(M)$ with the additive group $\square^\omega \mathbb{R}$. The latter space contains the clopen subgroup

$$c_0 = \left\{ (a_n)_{n \in \omega} \in \square^\omega \mathbb{R} : \lim_{n \rightarrow \infty} a_n = 0 \right\}.$$

For any $f \in \mathcal{H}(M) \setminus \mathcal{H}_c(M)$, we can find a countable discrete subset $X = \{x_n\}_{n \in \omega}$ of M such that $f(X) \cap X = \emptyset$. Indeed, the set $F = \text{supp}(f)$ is non-compact and closed in M , whence F is paracompact.

²It is proved in [13] that $\mathcal{H}(X)$ is a topological group if X is metrizable.

Since the compactness coincides with the pseudocompactness in the class of paracompact spaces (see [12, 3.10.21, 5.1.20]), F is not pseudocompact and hence admits a continuous unbounded function which extends to a continuous function $\xi : M \rightarrow [0, +\infty)$ by the normality of M . Since $\xi|_F$ is unbounded, we can choose a countable subset $X = \{x_n\}_{n \in \omega}$ in F so that for each $n \in \omega$, $f(x_n) \neq x_n$, $\xi(x_n) > n$ and

$$\xi(x_n) > \max \{ \xi(x_i), \xi(f(x_i)), \xi(f^{-1}(x_i)) : i < n \}.$$

Then $f(X) \cap X = \emptyset$ and $\lim_{n \rightarrow \infty} \xi(x_n) = \infty$.

By the normality of M , there exists a Urysohn map $\lambda : M \rightarrow [0, 1]$ with $\lambda(X) = 0$ and $\lambda(f(X)) = 1$. Since $h(X)$ is discrete for each $h \in \mathcal{H}(M)$, we have the map $\varphi : \mathcal{H}(M) \rightarrow \square^\omega \mathbb{R}$ defined by $\varphi(h) = (\lambda(h(x_n)))_{n \in \omega}$. Since $\varphi(\text{id}_M) = (0, 0, \dots)$ and $\varphi(f) = (1, 1, \dots)$, it follows that $\mathcal{U} \equiv \varphi^{-1}(c_0)$ is a clopen neighborhood of id_M with $f \notin \mathcal{U}$.

(2) Given a compact subset $\mathcal{K} \subset \mathcal{H}_c(M)$ we shall show that $\text{supp}(\mathcal{K}) = \text{cl}_M(\bigcup_{h \in \mathcal{K}} \text{supp}(h))$ is compact. Assume conversely that the set $\text{supp}(\mathcal{K})$ is not compact. The same argument as in (i) yields a continuous function $\xi : M \rightarrow [0, \infty)$ whose restriction $\xi|_{\text{supp}(\mathcal{K})}$ is unbounded. By induction we can choose sequences of points $x_n \in \text{supp}(\mathcal{K})$ and of homeomorphisms $h_n \in \mathcal{K}$ ($n \in \omega$) such that $h_n(x_n) \neq x_n$, $\xi(x_n) > n$ and

$$\xi(x_n) > \max \xi(\bigcup_{i < n} \text{supp}(h_i)).$$

It is seen that $x_n \in \text{supp}(h_n) \setminus \bigcup_{i < n} \text{supp}(h_i)$ and so $x_n \neq x_m$ if $n \neq m$.

By the compactness of \mathcal{K} , the sequence $(h_i)_{i \in \mathbb{N}}$ has a cluster point $h_\infty \in \mathcal{K} \subset \mathcal{H}_c(M)$. Note that $U_0 = M \setminus \{x_n\}_{n \in \omega}$ is open in M . For each $n \in \omega$, take a small open neighborhood U_n of x_n in M such that $h_n(x_n) \notin U_n$ and $U_n \cap U_m = \emptyset$ if $n \neq m$. Then we have $\mathcal{U} = \{U_n\}_{n \in \omega} \in \text{cov}(M)$. The neighborhood $\mathcal{U}(h_\infty)$ of h_∞ contains only finitely many h_n because $h_\infty(x_n) = x_n \in U_n$ for sufficiently large $n \in \omega$, but $h_n(x_n) \notin U_n$ for each $n \in \omega$. This contradicts the choice of h_∞ as a cluster point of $(h_n)_{n \in \omega}$. \square

Considering Conjecture in Introduction, we are concerned with the paracompactness of the space $\mathcal{H}_c(M)$.

Proposition 3.4. *For a locally compact separable metrizable space M , the space $\mathcal{H}_c(M)$ is (strongly) paracompact.*

Proof. We can write $M = \bigcup_{n \in \mathbb{N}} M_n$, where each M_n is compact and $M_n \subset \text{int}_M M_{n+1}$. Then $\mathcal{H}_c(M) = \bigcup_{n \in \mathbb{N}} \mathcal{H}(M, M \setminus \text{int} M_n)$. For each $n \in \mathbb{N}$, $\mathcal{H}(M, M \setminus \text{int} M_n) \approx \mathcal{H}(M_n, \text{bd}_M M_n)$ is separable metrizable, hence Lindelöf. Therefore, $\mathcal{H}_c(M)$ is Lindelöf by Theorem 3.8.5 in [12], so it is (strongly) paracompact by Theorem 5.1.2 (Corollary 5.3.11) in [12]. \square

This proposition has a further refinement. Following E. Michael [25] (see also [16]) we define a regular topological space X to be an \aleph_0 -space if X possesses a countable family \mathcal{N} of subsets of X such that for each open subset U of X and a compact subset K in U there is a finite subfamily \mathcal{F} of \mathcal{N} with $K \subset \bigcup \mathcal{F} \subset U$. Such a family \mathcal{N} is called a k -network for X . It is clear that each \aleph_0 -space has countable network weight (cf. [12, p.127]) and hence is Lindelöf [12, Theorem 3.8.12].

Proposition 3.5. *For a locally compact separable metrizable space M , the space $\mathcal{H}_c(M)$ is an \aleph_0 -space.*

Proof. In the proof of Proposition 3.4, each $\mathcal{H}(M, M \setminus \text{int} M_n)$ is separable metrizable and hence its topology has countable base \mathcal{B}_n . We claim that the union $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ is a countable k -network for $\mathcal{H}_c(M)$. Indeed, given an open set $U \subset \mathcal{H}_c(M)$ and a compact subset $K \subset U$, we can apply Proposition 3.3 (2) in order to

find $n \in \mathbb{N}$ such that $K \subset \mathcal{H}(M, M \setminus \text{int}M_n)$. The compactness of K allows us to find a finite subfamily \mathcal{F} of the base \mathcal{B}_n such that $K \subset \bigcup \mathcal{F} \subset U$. \square

4. TRANSFORMATION GROUPS

4.1. Generalities on transformation groups.

Throughout the article, a *transformation group* G on a space M means a topological group G acting on M continuously and effectively. Each $g \in G$ induces a homeomorphism of M , which is also denoted by the same symbol g . This determines the canonical injection $G \hookrightarrow \mathcal{H}(M)$. Let G_0 denote the connected component of the identity element e in G and let $G_c = \{g \in G : \text{supp}(g) \text{ is compact}\}$. For any subsets K, N of M , we obtain the following subgroups of G :

$$G_K = \{g \in G : g|_K = \text{id}_K\}, \quad G(N) = G_{M \setminus N}, \quad G_K(N) = G_K \cap G(N), \quad G_{K,c} = G_K \cap G_c.$$

Proposition 4.1. *Suppose the natural injection $G \rightarrow \mathcal{H}(M)$ is continuous.*

- (1) *If M is paracompact, then $G_0 \subset G_c$ and every compact subspace $\mathcal{K} \subset G_c$ is contained in $G(K)$ for some compact subset $K \subset M$.*
- (2) *If M is locally compact and σ -compact and $G(K)$ is second countable for every compact subset $K \subset M$, then G_c is an \aleph_0 -space (hence (strongly) paracompact).*

Proof. The statement (1) follows immediately from Proposition 3.3, while (2) can be deduced from (1) by analogy of the proof of Proposition 3.5. \square

For any subgroup H of G , there is a natural projection $\pi : G \rightarrow G/H$. The coset space $G/H = \{gH : g \in G\}$ is endowed with the quotient topology. The left coset $\pi(g) = gH \in G/H$ is also denoted by \bar{g} . The symbols H_0 and $(G/H)_0$ denote the connected components of e in H and \bar{e} in G/H respectively.

For each subset $L \subset M$, let $\mathcal{E}^G(L, M)$ be the set of embeddings $g|_L : L \rightarrow M$ induced by $g \in G$. The inclusion map $i_L : L \subset M$ is regarded as the distinguished point of this set. The group G acts transitively on the set $\mathcal{E}^G(L, M)$ by $g \cdot h = gh$. More generally, for subsets $K \subset L \subset N$ of M , we have the subset

$$\mathcal{E}_K^G(L, N) = \{g|_L \in \mathcal{E}(L, M) : g \in G_K(N)\} = G_K(N) \cdot i_L \subset \mathcal{E}^G(L, M).$$

The subgroup $G_K(N)$ acts on this subset transitively and the subgroup $G_L(N)$ is the stabilizer of i_L under this action. Since for each $g, g' \in G_K(N)$,

$$g|_L = g'|_L \iff g^{-1}g' \in G_L(N) \iff gG_L(N) = g'G_L(N),$$

the restriction map $r : G_K(N) \rightarrow \mathcal{E}_K^G(L, N)$, $r(g) = g|_L$, has the following factorization:

$$\begin{array}{ccc} & G_K(N) & \\ \pi \swarrow & & \searrow r \\ G_K(N)/G_L(N) & \xrightarrow[\text{bij.}]{\phi} & \mathcal{E}_K^G(L, N), \end{array}$$

where $\phi(\bar{g}) = g|_L = g \cdot i_L$. The map ϕ is a $G_K(N)$ -equivariant bijection. For $K \subset L_2 \subset L_1 \subset N$, we obtain the restriction map $\mathcal{E}_K^G(L_1, N) \rightarrow \mathcal{E}_K^G(L_2, N)$. Hereafter, in case $K = \emptyset$, the symbol K is omitted from the notations.

Here we include general remarks on topologies on the set $\mathcal{E}_K^G(L, N)$.

Definition 4.2. A topology τ on the set $\mathcal{E}_K^G(L, N)$ is said to be *admissible* if the action of $G_K(N)$ on $(\mathcal{E}_K^G(L, N), \tau)$ is continuous. The space $(\mathcal{E}_K^G(L, N), \tau)$ is also denoted by $\mathcal{E}_K^G(L, N)^\tau$.

Let $\hat{\tau} = \hat{\tau}_K^G(L, N)$ denote the quotient topology on $\mathcal{E}_K^G(L, N)$ induced by the map r . For simplicity, the space $(\mathcal{E}_K^G(L, N), \hat{\tau})$ is denoted by $\hat{\mathcal{E}}_K^G(L, N)$.

Remark 4.3. (i) The quotient topology $\hat{\tau}$ is the strongest admissible topology on $\mathcal{E}_K^G(L, N)$.

(ii) The restriction map $\hat{\mathcal{E}}_{K_1}^G(L_1, N_1) \rightarrow \hat{\mathcal{E}}_{K_2}^G(L_2, N_2)$ is continuous for triples (N_1, L_1, K_1) and (N_2, L_2, K_2) such that $N_1 \subset N_2$, $L_1 \supset L_2$ and $K_1 \supset K_2$.

Remark 4.4. If the set $\mathcal{E}_K^G(L, N)$ is equipped with an admissible topology τ , then the following hold:

- (i) The maps r and ϕ are continuous. The map ϕ is a homeomorphism if and only if $\tau = \hat{\tau}$.
- (ii) The map $r : G_K(N) \rightarrow \mathcal{E}_K^G(L, N)^\tau$ has a local section at i_L if and only if $\tau = \hat{\tau}$ and the map

$$\pi : G_K(N) \rightarrow G_K(N)/G_L(N)$$

has a local section. In this case, the map r is a principal $G_L(N)$ -bundle.

4.2. Local section property.

Throughout this subsection, we assume that G is a transformation group on a space M . Suppose $K \subset L \subset N$ are subsets of M .

Definition 4.5. We say that the triple (N, L, K) has the *local section property* for G (abbrev. LSP_G) if the restriction map $r : G_K(N) \rightarrow (\mathcal{E}_K^G(L, M), \tau)$ has a local section s at the inclusion $i_L : L \subset M$ with respect to some admissible topology τ . (In this definition, one should not confuse $\mathcal{E}_K^G(L, M)$ with $\mathcal{E}_K^G(L, N)$.)

Remark 4.6. (0) The above map s is also a local section of the restriction map $G_K \rightarrow \mathcal{E}_K^G(L, M)^\tau$. Thus $\tau = \hat{\tau}_K^G(L, M)$ by Remark 4.4 (ii).

- (i) We can modify the local section s so that $s(i_L) = \text{id}_M$.
- (ii) A triple (N, L, K) has LSP_G if and only if for some admissible topology τ
 - (a) $\mathcal{E}_K^G(L, N)$ is open in $\mathcal{E}_K^G(L, M)^\tau$ and
 - (b) the map $r : G_K(N) \rightarrow \mathcal{E}_K^G(L, N)^\tau$ has a local section at i_L .

(iii) Suppose (N_1, L, K_1) and (N_2, L, K_2) are triples of subsets of M such that $N_1 \subset N_2$ and $K_1 \subset K_2$. If (N_1, L, K_1) has LSP_G , then so does (N_2, L, K_2) . Indeed, the map $r_1 : G_{K_1}(N_1) \rightarrow \mathcal{E}_{K_1}^G(L, M)^\tau$ has a local section $s : \mathcal{U} \rightarrow G_{K_1}(N_1)$ at the inclusion $i_L : L \subset M$. For each $f \in \mathcal{U} \cap \mathcal{E}_{K_2}^G(L, M)^\tau$, since $s(f)|_L = f$ and $K_2 \subset L$, we have $s(f)|_{K_2} = f|_{K_2} = \text{id}$, hence the restriction of s is a local section for $r_2 : G_{K_2}(N_2) \rightarrow \mathcal{E}_{K_2}^G(L, M)^\tau$.

Definition 4.7. Suppose τ is an admissible topology on $\mathcal{E}_K^G(N, M)$. We say that the triple (N, L, K) has the *weak local section property* for G with respect to τ (abbrev. $\text{WLSP}_{G, \tau}$) if there exists an open neighborhood \mathcal{V} of i_N in $(\mathcal{E}_K^G(N, M), \tau)$ and a continuous map $s : \mathcal{V} \rightarrow G_K(N)$ such that $s(f)|_L = f|_L$ for each $f \in \mathcal{V}$.

Remark 4.8. (i) We can modify the map s so that $s(i_N) = \text{id}_M$.

- (ii) When the topology τ is understood from the context, we omit the symbol τ from the notations.

4.3. Exhausting sequences.

This subsection includes a remark on exhausting sequences of spaces and their associated towers in transformation groups. Suppose M is a locally compact σ -compact space and G is a transformation group on M . Recall that a subset A of M is *regular closed* if $A = \text{cl}_M(\text{int}_M A)$ and that a sequence $\mathcal{F} = (F_i)_{i \in \mathbb{N}}$ of subsets of M is *discrete* if each point $x \in M$ has a neighborhood which meets at most one F_i .

By the assumption, there exists a sequence $(M_i)_{i \in \mathbb{N}}$ of compact regular closed sets in M such that $M_i \subset \text{int}_M M_{i+1}$ and $M = \bigcup_{i \in \mathbb{N}} M_i (= \bigcup_{i \in \mathbb{N}} \text{int}_M M_i)$. It induces the tower $(G(M_i))_{i \in \mathbb{N}}$ of closed subgroups of G_c and the multiplication map

$$p : \square_{i \in \mathbb{N}} G(M_i) \rightarrow G_c, \quad p(h_1, \dots, h_n) = h_1 \cdots h_n.$$

For each $i \in \mathbb{N}$, let $K_i = M \setminus \text{int}_M M_i$ and $L_i = M_i \setminus \text{int}_M M_{i-1}$, where $M_0 = \emptyset$. Then the sequences $(L_{2i-1})_{i \in \mathbb{N}}$ and $(L_{2i})_{i \in \mathbb{N}}$ are discrete in M . There exists a sequence $(N_i)_{i \in \mathbb{N}}$ of compact subsets of M such that $L_i \subset \text{int}_M N_i$ and $N_i \cap N_j = \emptyset$ if $|i-j| \geq 2$ (hence $N_i \subset \text{int}_M M_{i+1} \setminus M_{i-2}$ and subsequences $(N_{2i-1})_{i \in \mathbb{N}}$ and $(N_{2i})_{i \in \mathbb{N}}$ are discrete in M). Note that $G(M_i) = G_{K_i}$ and L_i is regular closed since $L_i = \text{cl}_M((\text{int}_M M_i) \setminus M_{i-1})$. We call each of the sequences $(M_i)_{i \in \mathbb{N}}$, $(M_i, L_i, N_i)_{i \in \mathbb{N}}$ and $(M_i, K_i, L_i, N_i)_{i \in \mathbb{N}}$ an *exhausting sequence* for M .

The next lemma directly follows from Lemma 2.11.

Lemma 4.9. *Suppose $(M_i)_{i \in \mathbb{N}}$ is an exhausting sequence for M and the map $p : \square_{i \in \mathbb{N}} G(M_i) \rightarrow G_c$ has a local section at the identity element e . Then the following hold:*

- (1) *The map p has a local section at any point of G_c .*
- (2) *If each $G(M_i)$ is locally contractible, then so is G_c .*
- (3) *If G_c is paracompact, H is a subgroup of G and each $H(M_i)$ is HD in $G(M_i)$, then H_c is HD in G_c .*

5. TRANSFORMATION GROUPS WITH STRONG TOPOLOGY

In Section 6, we shall study the topological properties of the homeomorphism group $\mathcal{H}(M)$ and the diffeomorphism group $\mathcal{D}(M)$ of a non-compact manifold M endowed with the Whitney topology. These groups acts naturally on the manifold M and admit infinite products of elements with discrete supports. This geometric property distinguishes these groups from other abstract topological groups and connects them to the box topology. To clarify this situation, we introduce the notion of transformation groups with strong topology and study its basic properties. In particular, we obtain some conditions under which transformation groups are locally homeomorphic to some box/small box products (Propositions 5.5 and 5.11). In Section 6, these fundamental results are applied to yield main results of this article. Our axiomatic approach is also intended for further applications to the study of other subgroups of $\mathcal{H}(M)$ and $\mathcal{D}(M)$.

5.1. Transformation groups with strong topology.

Throughout this subsection, let G be a transformation group on a space M . For each discrete sequence $\mathcal{L} = (L_i)_{i \in \mathbb{N}}$ of subsets of M , we have

- (i) the group homomorphism $\lambda_{\mathcal{L}} : \square_{i \in \mathbb{N}} G(L_i) \rightarrow \mathcal{H}(M)$ defined by

$$\lambda_{\mathcal{L}}((g_i)_{i \in \mathbb{N}})|_{L_j} = g_j|_{L_j} \quad (j \in \mathbb{N}) \quad \text{and} \quad \lambda_{\mathcal{L}}((g_i)_{i \in \mathbb{N}})|_{M \setminus \bigcup_{i \in \mathbb{N}} L_i} = \text{id}.$$

- (ii) the function $r_{\mathcal{L}} : G \rightarrow \square_{i \in \mathbb{N}} \mathcal{E}^G(L_i, M)$ defined by

$$r_{\mathcal{L}}(g) = (g|_{L_i})_{i \in \mathbb{N}}.$$

Note that $\lambda_{\mathcal{L}}^{-1}(\mathcal{H}_c(M)) = \square_{i \in \mathbb{N}} G(L_i)$ if each L_i is compact.

Definition 5.1. We say that the transformation group G on M has a *strong topology* if it satisfies the following conditions:

- (1) The natural injection $G \rightarrow \mathcal{H}(M)$ is continuous with respect to the Whitney topology on $\mathcal{H}(M)$.
- (2) For any discrete sequence $\mathcal{L} = (L_i)_{i \in \mathbb{N}}$ in M ,
 - (i) $\text{im } \lambda_{\mathcal{L}} \subset G$ and
 - (ii) the map $\lambda_{\mathcal{L}} : \square_{i \in \mathbb{N}} G(L_i) \rightarrow G(\bigcup_{i \in \mathbb{N}} L_i)$ is an open embedding.

Definition 5.2. Let \mathcal{F} be a collection of subsets of M . We say that the transformation group G on M has a *strong topology with respect to \mathcal{F}* if G has a strong topology and satisfies the following additional condition:

- (*) For any discrete sequence $\mathcal{L} = (L_i)_{i \in \mathbb{N}}$ in M with $L_i \in \mathcal{F}$, the function $r_{\mathcal{L}} : G \rightarrow \square_{i \in \mathbb{N}} \widehat{\mathcal{E}}^G(L_i, M)$ is continuous.

5.2. Transformation groups locally homeomorphic to box products.

Assumption 5.3. Throughout this subsection, we assume that M is a locally compact σ -compact space and G is a transformation group on M with a strong topology with respect to a collection \mathcal{F} of subsets of M .

For notational simplicity, for pairs of spaces and maps between them, we use the following terminology and notations: For pairs of spaces (X, A) and (Y, B) , we set $(X, A) \times (Y, B) = (X \times Y, A \times B)$. We say that

- (i) (X, A) and (Y, B) are *locally homeomorphic* and write $(X, A) \approx_{\ell} (Y, B)$ if for each point $a \in A$ there exists an open neighborhood U of a in X and an open subset V of Y which admit a homeomorphism of pairs of spaces $(U, U \cap A) \approx (V, V \cap B)$.
- (ii) a map $p : (X, A) \rightarrow (Y, B)$ of pairs of spaces has a *local section* at a point $b \in B$, if there exists an open neighborhood V of b in Y and a map $s : (V, V \cap B) \rightarrow (X, A)$ of pairs such that $ps = \text{id}_V$.

Suppose $\mathcal{L} = (L_i)_{i \in \mathbb{N}}$, $\mathcal{N} = (N_i)_{i \in \mathbb{N}}$ and $\mathcal{K} = (K_i)_{i \in \mathbb{N}}$ are discrete sequences of compact subsets of M such that $L_i \subset N_i$ for each $i \in \mathbb{N}$ and $M = L \cup K$, where $L = \bigcup_{i \in \mathbb{N}} L_i$ and $K = \bigcup_{i \in \mathbb{N}} K_i$. Since G has a strong topology, the discrete sequences \mathcal{N} and \mathcal{K} induce the open embeddings

$$\lambda_{\mathcal{N}} : \square_{i \in \mathbb{N}} G(N_i) \rightarrow G(N) \quad \text{and} \quad \lambda_{\mathcal{K}} : \square_{i \in \mathbb{N}} G(K_i) \rightarrow G(K),$$

where $N = \bigcup_{i \in \mathbb{N}} N_i$. Then we obtain the map

$$\rho : (\square, \square)_{i \in \mathbb{N}} G(N_i) \times (\square, \square)_{i \in \mathbb{N}} G(K_i) \rightarrow (G, G_c),$$

defined by $\rho((g_i)_{i \in \mathbb{N}}, (h_i)_{i \in \mathbb{N}}) = \lambda_{\mathcal{N}}((g_i)_{i \in \mathbb{N}}) \lambda_{\mathcal{K}}((h_i)_{i \in \mathbb{N}})$.

Lemma 5.4. *If (N_i, L_i) has LSP_G and $L_i \in \mathcal{F}$ for each $i \in \mathbb{N}$, then*

- (1) $(G, G_c) \approx_{\ell} (\square, \square)_{i \in \mathbb{N}} \widehat{\mathcal{E}}^G(L_i, M) \times (\square, \square)_{i \in \mathbb{N}} G(K_i)$,
- (2) *the map ρ has a local section at id_M .*

Proof. We are concerned with the following maps:

$$\begin{aligned} \mathcal{V} &\xrightarrow{r_{\mathcal{L}}} \square_{i \in \mathbb{N}} \mathcal{V}_i \xrightarrow{s} \square_{i \in \mathbb{N}} G(N_i) \xrightarrow{\lambda_{\mathcal{N}}} G(N), \\ \mathcal{V} &\xrightarrow{\phi} \square_{i \in \mathbb{N}} \mathcal{V}_i \times G_L \xrightarrow{s \times \text{id}} \square_{i \in \mathbb{N}} G(N_i) \times G_L \xrightarrow{\theta} G, \\ \eta &= \lambda_{\mathcal{N}} s r_{\mathcal{L}}, \quad \theta(s \times \text{id}) \phi = \text{id}_{\mathcal{V}}, \quad \psi = \theta(s \times \text{id}). \end{aligned}$$

These are defined as follows: Since G has a strong topology with respect to \mathcal{F} and $L_i \in \mathcal{F}$ ($i \in \mathbb{N}$), we obtain the map

$$r_{\mathcal{L}} : G \rightarrow \square_{i \in \mathbb{N}} \widehat{\mathcal{E}}^G(L_i, M), \quad r_{\mathcal{L}}(g) = (g|_{L_i})_{i \in \mathbb{N}}.$$

By the assumption, for each $i \in \mathbb{N}$, there exists an open neighborhood \mathcal{V}_i of the inclusion $i_{L_i} : L_i \subset M$ in $\widehat{\mathcal{E}}^G(L_i, M)$ and a map $s_i : \mathcal{V}_i \rightarrow G(N_i)$ such that $s_i(f)|_{L_i} = f|_{L_i}$ for each $f \in \mathcal{V}_i$ and $s_i(i_{L_i}) = \text{id}_M$. The maps s_i ($i \in \mathbb{N}$) determine the map

$$s : \square_{i \in \mathbb{N}} \mathcal{V}_i \rightarrow \square_{i \in \mathbb{N}} G(N_i), \quad s((f_i)_{i \in \mathbb{N}}) = (s_i(f_i))_{i \in \mathbb{N}}.$$

The preimage $\mathcal{V} = r_{\mathcal{L}}^{-1}(\square_{i \in \mathbb{N}} \mathcal{V}_i)$ is an open neighborhood of id_M in G . Let $\eta = \lambda_{\mathcal{N}} s r_{\mathcal{L}} : \mathcal{V} \rightarrow G(N)$. For every $g \in \mathcal{V}$, since $\eta(g)|_{L_i} = g|_{L_i}$ for each $i \in \mathbb{N}$, we have $\eta(g)^{-1}g \in G_L$. The maps ϕ and θ are defined by

$$\phi(g) = (r_{\mathcal{L}}(g), \eta(g)^{-1}g) \quad \text{and} \quad \theta((g_i)_{i \in \mathbb{N}}, h) = \lambda_{\mathcal{N}}((g_i)_{i \in \mathbb{N}})h.$$

Then $\theta(s \times \text{id})\phi = \text{id}_{\mathcal{V}}$ because $\theta(s \times \text{id})\phi(g) = \lambda_{\mathcal{N}} s r_{\mathcal{L}}(g) \eta(g)^{-1}g = g$ for each $g \in \mathcal{V}$. Since $M = L \cup K$, we have $G_L \subset G(K)$. It follows from the definition of $\lambda_{\mathcal{K}}$ that

$$(\square, \square)_{i \in \mathbb{N}} G_L(K_i) \xrightarrow{\lambda_{\mathcal{K}}} (\text{im } \lambda_{\mathcal{K}} \cap G_L, \text{im } \lambda_{\mathcal{K}} \cap G_{L,c}) \approx_{\ell} (G_L, G_{L,c}).$$

(1) Consider the map $\psi = \theta(s \times \text{id})$. For each $((f_i)_{i \in \mathbb{N}}, h) \in \square_{i \in \mathbb{N}} \mathcal{V}_i \times G_L$, we can write

$$\psi((f_i)_{i \in \mathbb{N}}, h) = fh, \quad \text{where} \quad f = \lambda_{\mathcal{N}}((s_i(f_i))_{i \in \mathbb{N}}).$$

Since $fh|_{L_i} = s_i(f_i)|_{L_i} = f_i \in \mathcal{V}_i$, it follows that

$$r_{\mathcal{L}}(fh) = (f_i)_{i \in \mathbb{N}} \in \square_{i \in \mathbb{N}} \mathcal{V}_i,$$

which means $fh \in \mathcal{V}$. Thus, we obtain the map

$$\psi : \square_{i \in \mathbb{N}} \mathcal{V}_i \times G_L \rightarrow \mathcal{V}$$

such that $\psi\phi = \theta(s \times \text{id})\phi = \text{id}_{\mathcal{V}}$. Moreover,

$$\eta(fh)^{-1}fh = \lambda_{\mathcal{N}}((s(f_i))_{i \in \mathbb{N}})^{-1}fh = h.$$

Since $r_{\mathcal{L}}(fh) = (f_i)_{i \in \mathbb{N}}$, it follows that $\phi\psi((f_i)_{i \in \mathbb{N}}, h) = ((f_i)_{i \in \mathbb{N}}, h)$. Therefore, ϕ is a homeomorphism with $\phi^{-1} = \psi$. On the other hand, $\phi(\mathcal{V} \cap G_c) = \square_{i \in \mathbb{N}} \mathcal{V}_i \times G_{L,c}$ because $s_i(i_{L_i}) = \text{id}_M$. Consequently,

$$(\mathcal{V}, \mathcal{V} \cap G_c) \approx (\square, \square)_{i \in \mathbb{N}} \mathcal{V}_i \times (G_L, G_{L,c}).$$

Recall $(G_L, G_{L,c}) \approx_{\ell} (\square, \square)_{i \in \mathbb{N}} G_L(K_i)$. Thus, we have

$$(G, G_c) \approx_{\ell} (\square, \square)_{i \in \mathbb{N}} \widehat{\mathcal{E}}^G(L_i, M) \times (\square, \square)_{i \in \mathbb{N}} G_L(K_i).$$

(2) The map ρ has the factorization

$$\begin{array}{ccc} \square_{i \in \mathbb{N}} G(N_i) \times \square_{i \in \mathbb{N}} G_L(K_i) & \xrightarrow{\rho} & G \\ \text{id} \times \lambda_{\mathcal{K}} \downarrow \approx & & \uparrow \theta \\ \square_{i \in \mathbb{N}} G(N_i) \times (\text{im } \lambda_{\mathcal{K}} \cap G_L) & \xrightarrow{\subset} & \square_{i \in \mathbb{N}} G(N_i) \times G_L \end{array}$$

Since $\theta(s \times \text{id})\phi = \text{id}_{\mathcal{V}}$, the map θ has the next local section at id_M :

$$\sigma_0 = (s \times \text{id})\phi : (\mathcal{V}, \mathcal{V} \cap G_c) \rightarrow (\square, \square)_{i \in \mathbb{N}} G(N_i) \times (G_L, G_{L,c}).$$

Since $\text{im}(\text{id} \times \lambda_K)$ is an open neighborhood of

$$\sigma_0(\text{id}_M) = ((\text{id}_M, \text{id}_M, \dots), \text{id}_M)$$

in $\square_{i \in \mathbb{N}} G(N_i) \times G_L$, we have a neighborhood \mathcal{U} of id_M in \mathcal{V} such that $\sigma_0(\mathcal{U}) \subset \text{im}(\text{id} \times \lambda_K)$. The following map is a local section of ρ at id_M :

$$\sigma = (\text{id} \times \lambda_K^{-1}) \sigma_0|_{\mathcal{U}} : (\mathcal{U}, \mathcal{U} \cap G_c) \rightarrow (\square, \square)_{i \in \mathbb{N}} G(N_i) \times (\square, \square)_{i \in \mathbb{N}} G_L(K_i).$$

This completes the proof. \square

Proposition 5.5. *Suppose $(M_i, L_i, N_i)_{i \in \mathbb{N}}$ is an exhausting sequence for M . If (N_{2i}, L_{2i}) has LSP_G and $L_{2i} \in \mathcal{F}$ for each $i \in \mathbb{N}$, then the following hold:*

- (1) $(G, G_c) \approx_\ell (\square, \square)_{i \in \mathbb{N}} \widehat{\mathcal{E}}^G(L_{2i}, M) \times (\square, \square)_{i \in \mathbb{N}} G(L_{2i-1})$,
- (2) *The multiplication map $p : \square_{i \in \mathbb{N}} G(M_i) \rightarrow G_c$ has a local section at any point of G_c .*

Proof. We apply Lemma 5.4 to the discrete families $\mathcal{L} = (L_{2i})_{i \in \mathbb{N}}$, $\mathcal{N} = (N_{2i})_{i \in \mathbb{N}}$ and $\mathcal{K} = (L_{2i-1})_{i \in \mathbb{N}}$. Let $L = \bigcup_{i \in \mathbb{N}} L_{2i}$. Then $G_L(L_{2i-1}) = G(L_{2i-1})$ for each $i \in \mathbb{N}$ because L_{2i-2} is regular closed. The statement (1) is none other than Lemma 5.4 (1). It remains to show the statement (2). Due to Lemma 5.4 (2), the map

$$\begin{aligned} \rho : \square_{i \in \mathbb{N}} G(N_{2i}) \times \square_{i \in \mathbb{N}} G(L_{2i-1}) &\longrightarrow G_c, \\ \rho((f_i)_{i \in \mathbb{N}}, (g_i)_{i \in \mathbb{N}}) &= \lambda_{\mathcal{N}}((f_i)_{i \in \mathbb{N}}) \lambda_{\mathcal{K}}((g_i)_{i \in \mathbb{N}}) \end{aligned}$$

has a local section at id_M , say

$$\sigma : \mathcal{W} \longrightarrow \square_{i \in \mathbb{N}} G(N_{2i}) \times \square_{i \in \mathbb{N}} G(L_{2i-1}).$$

Note that the image $\sigma(h) = ((f_i)_{i \in \mathbb{N}}, (g_i)_{i \in \mathbb{N}})$ of each $h \in \mathcal{W}$ satisfies the following conditions:

- (a) $h = \lambda_{\mathcal{N}}((f_i)_{i \in \mathbb{N}}) \lambda_{\mathcal{K}}((g_i)_{i \in \mathbb{N}}) = (f_1 f_2 \dots)(g_1 g_2 \dots) = f_1 g_1 f_2 g_2 f_3 g_3 \dots$;
- (b) $f_i \in G(N_{2i}) \subset G(M_{2i+1})$ and $g_i \in G(L_{2i-1}) \subset G(M_{2i-1}) \subset G(M_{2i+2})$ for each $i \in \mathbb{N}$;
- (c) $(\text{id}_M, \text{id}_M, f_1, g_1, f_2, g_2, \dots) \in \square_{i \in \mathbb{N}} G(M_i)$ and $h = p(\text{id}_M, \text{id}_M, f_1, g_1, f_2, g_2, \dots)$.

Therefore, a local section at id_M of the map $p : \square_{i \in \mathbb{N}} G(M_i) \rightarrow G_c$ is defined by

$$s : \mathcal{W} \rightarrow \square_{i \in \mathbb{N}} G(M_i), \quad s(h) = (\text{id}_M, \text{id}_M, f_1, g_1, f_2, g_2, \dots).$$

The conclusion now follows from Lemma 4.9 (1). This completes the proof. \square

5.3. Strong topology with respect to an admissible collection.

In the cases where G is the diffeomorphism group of a smooth n -manifold ($n \geq 1$) or the homeomorphism group of a topological n -manifold ($n = 1, 2$), we can apply Proposition 5.5. However, when G is the homeomorphism group of an n -manifold M ($n \geq 3$), it is still an open problem whether the restriction map $r : G \rightarrow \mathcal{E}^G(L, M)$ has a local section for a locally flat n -submanifold L of M (cf. Subsection 6.1). At this moment, we only know that the deformation theorem for embeddings in topological manifolds ([11]) implies the weak local section property. This motivates the formulation of this section.

Throughout this subsection, we assume that G is a transformation group on a space M .

Definition 5.6. An *admissible collection* $(\mathcal{F}, \tau^G(*, M))$ for the transformation group G on M consists of a collection \mathcal{F} of subsets of M and an assignment of an admissible topology $\tau^G(L, M)$ on the set $\mathcal{E}^G(L, M)$ to each $L \in \mathcal{F}$.

Convention 5.7. When an admissible collection $(\mathcal{F}, \tau^G(*, M))$ is fixed, for any triple (N, L, K) of subsets of M with $L \in \mathcal{F}$, let $\tau_K^G(L, N)$ denote the subspace topology on $\mathcal{E}_K^G(L, N)$ inherited from the space $(\mathcal{E}^G(L, M), \tau^G(L, M))$, which is an admissible topology on $\mathcal{E}_K^G(L, N)$.

Definition 5.8. Let $(\mathcal{F}, \tau^G(*, M))$ be an admissible collection for the transformation group G on M . We say that G has a *strong topology* with respect to $(\mathcal{F}, \tau^G(*, M))$ if G has a strong topology and satisfies the following additional condition:

(**) For any discrete sequence $\mathcal{L} = (L_i)_{i \in \mathbb{N}}$ in M with $L_i \in \mathcal{F}$, the following function is continuous:

$$r_{\mathcal{L}} : G \rightarrow \square_{i \in \mathbb{N}}(\mathcal{E}^G(L_i, M), \tau^G(L_i, M)).$$

Assumption 5.9. Below we assume that M is a locally compact σ -compact space and G is a transformation group on M with a strong topology with respect to an admissible collection $(\mathcal{F}, \tau^G(*, M))$.

Suppose $\mathcal{L} = (L_i)_{i \in \mathbb{N}}$, $\mathcal{N} = (N_i)_{i \in \mathbb{N}}$ and $\mathcal{K} = (K_i)_{i \in \mathbb{N}}$ are discrete sequences of subsets of M such that $L_i \subset N_i$ ($i \in \mathbb{N}$) and $M = L \cup K$. Here, $L = \bigcup_{i \in \mathbb{N}} L_i$, $N = \bigcup_{i \in \mathbb{N}} N_i$ and $K = \bigcup_{i \in \mathbb{N}} K_i$. Since G has a strong topology, the sequences \mathcal{N} and \mathcal{K} induce the open embeddings

$$\lambda_{\mathcal{N}} : \square_{i \in \mathbb{N}} G(N_i) \rightarrow G(N) \quad \text{and} \quad \lambda_{\mathcal{K}} : \square_{i \in \mathbb{N}} G(K_i) \rightarrow G(K).$$

These maps determine the map

$$\rho : (\square, \square)_{i \in \mathbb{N}} G(N_i) \times (\square, \square)_{i \in \mathbb{N}} G(K_i) \rightarrow (G, G_c), \quad \rho((g_i)_{i \in \mathbb{N}}, (h_i)_{i \in \mathbb{N}}) = \lambda_{\mathcal{N}}((g_i)_{i \in \mathbb{N}}) \lambda_{\mathcal{K}}((h_i)_{i \in \mathbb{N}}).$$

Under the weaker condition $WLSP_G$, we obtain the following conclusions.

Lemma 5.10. *If (N_i, L_i) has $WLSP_G$ and $N_i \in \mathcal{F}$ for each $i \in \mathbb{N}$, then the map ρ has a local section at id_M .*

Proof. The proof is exactly a repetition of the arguments in Lemma 5.4 (except the part (1)). There is only one point to be modified:

(†) The map $r_{\mathcal{L}}$ is replaced by the map $r_{\mathcal{N}}$ so that \mathcal{V}_i is an open neighborhood of the inclusion $i_{N_i} : N_i \subset M$ in the space $(\mathcal{E}^G(N_i, M), \tau^G(N_i, M))$.

The remaining parts are unchanged. □

Proposition 5.11. *Suppose $(M_i, L_i, N_i)_{i \in \mathbb{N}}$ is an exhausting sequence for M . If each (N_{2i}, L_{2i}) has $WLSP_G$ and $N_{2i} \in \mathcal{F}$, then the multiplication map $p : \square_{i \in \mathbb{N}} G(M_i) \rightarrow G_c$ has a local section at any point of G_c .*

Proof. The proof is completely same as that of Proposition 5.5 (2), except that we apply Lemma 5.10 instead of Lemma 5.4 (2). □

6. HOMEOMORPHISM AND DIFFEOMORPHISM GROUPS OF NON-COMPACT MANIFOLDS

In this section, we apply Propositions 5.5 and 5.11 in Section 5 to study local topological properties of homeomorphism groups and diffeomorphism groups of non-compact manifolds M endowed with the Whitney topology. For homeomorphism groups of topological n -manifolds M , in case $n = 2$ we can describe the local topological type of $\mathcal{H}(M)$ and $\mathcal{H}_c(M)$, while in case $n \geq 3$ the Homeomorphism Group Problem is still open and we are restricted only to show the local contractibility of $\mathcal{H}_c(M)$. On the other hand, for diffeomorphism groups of smooth n -manifolds, we can determine the local topological type of $\mathcal{D}(M)$ and $\mathcal{D}_c(M)$ in every dimension n .

6.1. Homeomorphism groups of non-compact n -manifolds.

Suppose M is a σ -compact topological n -manifold possibly with boundary. When M is compact, the group $\mathcal{H}(M)$ is known to be locally contractible ([10] and [11]). In this subsection, we apply Proposition 5.11 to extend this result to the noncompact case.

Proposition 6.1. *For every σ -compact n -manifold M possibly with boundary, the group $\mathcal{H}_c(M)$ is locally contractible.*

We follow the formulation in Subsection 5.3. The topological group $G = \mathcal{H}(M)$ admits the natural action on M . According to the convention of Subsection 4.1, we use the following notations for $K, N \subset M$:

$$\begin{aligned} \mathcal{H}_0(M) &= G_0, & \mathcal{H}_c(M) &= G_c, & \mathcal{H}(M, K) &= G_K, \\ \mathcal{H}(M, M \setminus N) &= G(N), & \mathcal{H}(M, K \cup (M \setminus N)) &= G_K(N), \\ \mathcal{H}_c(M, K) &= G_{K,c}, & \mathcal{H}_0(M, K) &= (G_K)_0. \end{aligned}$$

For subspaces $K \subset L \subset N \subset M$, the symbol $\mathcal{E}_K(L, N)$ denotes the space of embeddings $f : L \rightarrow N$ with $f|_K = \text{id}_K$ endowed with the compact-open topology (Section 3). Recall that an embedding $f : L \rightarrow M$ is *proper* if $f^{-1}(\partial M) = L \cap \partial M$. Let $\mathcal{E}_K^*(L, M)$ denote the subspace of $\mathcal{E}_K(L, M)$ consisting of all proper embeddings. Then $\mathcal{E}_K^G(L, M) \subset \mathcal{E}_K^*(L, M)$.

Let \mathcal{F} denote the collection of all subsets of M and $\mathcal{E}^G(L, M)$, $L \in \mathcal{F}$, be endowed with the compact-open topology $\tau^G(L, M)$, which is admissible, that is, the action of $G = \mathcal{H}(M)$ on $\mathcal{E}^G(L, M)$ is continuous. Then the transformation group G on M has a strong topology with respect to the admissible collection $(\mathcal{F}, \tau^G(*, M))$, that is, for each discrete sequence $\mathcal{L} = (L_i)_{i \in \mathbb{N}}$ in M (i) the map $\lambda_{\mathcal{L}} : \prod_{i \in \mathbb{N}} G(L_i) \rightarrow G(\bigcup_{i \in \mathbb{N}} L_i)$ is an open embedding and (ii) the map $r_{\mathcal{L}} : G \rightarrow \prod_{i \in \mathbb{N}} \mathcal{E}^G(L_i, M)$ is continuous with respect to the compact-open topology $\tau^G(L_i, M)$. Next, we shall inspect WLSP_G of compact subsets of M .

Lemma 6.2. *Suppose L is a compact subset of M and N is a compact neighborhood of L in M . Then the pair (N, L) has the WLSP_G (with respect to the topology $\tau^G(N, M)$), that is, there exists an open neighborhood \mathcal{V} of $i_N : N \subset M$ in $(\mathcal{E}^G(N, M), \tau^G(N, M))$ and a continuous map $s : \mathcal{V} \rightarrow G(N)$ such that $s(f)|_L = f|_L$ for each $f \in \mathcal{V}$.*

This lemma follows from the next version of the deformation theorem for topological embeddings [11].

Lemma 6.3. *Suppose $C \subset D$ are compact subsets of M with $C \subset \text{int}_M D$ and $K \subset L$ are closed subsets of M with $K \subset \text{int}_M L$. Then there exists an open neighborhood \mathcal{V} of the inclusion $i : D \cup L \subset M$ in $\mathcal{E}_L^*(D \cup L, M)$ and a map*

$$\phi : \mathcal{V} \rightarrow \mathcal{H}(M, K \cup (M \setminus \text{int}_M D))$$

such that $\phi(f)|_C = f|_C$ for each $f \in \mathcal{V}$ and $\phi(i) = \text{id}_M$.

Proof. Take a compact neighborhood E of C in $\text{int}_M D$. Then, by [11, Theorem 5.1] there exists an open neighborhood \mathcal{U} of the inclusion $i_D : D \subset M$ in $\mathcal{E}_{D \cap L}^*(D, M)$ and a map $\eta : \mathcal{U} \rightarrow \mathcal{E}_{D \cap K}^*(D, M)$ such that $\eta(i_D) = i_D$ and for each $f \in \mathcal{U}$

$$(a) \ \eta(f) = \text{id} \text{ on } E, \quad (b) \ \eta(f) = f \text{ on } \text{bd}_M D \quad \text{and} \quad (c) \ \eta(f)(D) = f(D).$$

Replacing \mathcal{U} by a smaller one, we may assume that $f(C) \subset E$. The required map ϕ is defined by $\phi(f)|_D = \eta(f|_D)^{-1}(f|_D)$ and $\phi(f) = \text{id}$ on $M \setminus D$. \square

Let $(M_i, K_i)_{i \in \mathbb{N}}$ be an exhausting sequence of M , that is, $M = \bigcup_{i \in \mathbb{N}} M_i$, each M_i is compact regular closed in M , $M_i \subset \text{int}_M M_{i+1}$ and $K_i = M \setminus \text{int}_M M_i$. Then $\mathcal{H}(M, K_i) = G_{K_i} = G(M_i)$ for each $i \in \mathbb{N}$. Consider the multiplication map

$$p : \square_{i \in \mathbb{N}} \mathcal{H}(M, K_i) \rightarrow \mathcal{H}_c(M), \quad p((h_i)_{i \in \mathbb{N}}) = h_1 h_2 h_3 \cdots.$$

Lemma 6.4. *The map $p : \square_{i \in \mathbb{N}} \mathcal{H}(M, K_i) \rightarrow \mathcal{H}_c(M)$ has a local section at id_M .*

Proof. We can form an exhausting sequence $(M_i, K_i, L_i, N_i)_{i \in \mathbb{N}}$, that is, $L_i = M_i \setminus \text{int}_M M_{i-1}$, $L_i \subset \text{int}_M N_i$ and $N_i \cap N_j = \emptyset$ if $|i - j| \geq 2$. Since each pair (N_i, L_i) has WLSP_G by Lemma 6.2, the conclusion follows from Proposition 5.11. \square

Proof of Proposition 6.1. By [1, Theorem 0] (cf. [21], [28]), there exists an exhausting sequence $(M_i)_{i \in \mathbb{N}}$ for M consisting of compact n -submanifolds of M such that $\text{bd}_M M_i$ is a compact proper $(n - 1)$ -submanifold of M which is transversal to ∂M . Since

$$G(M_i) = \mathcal{H}(M, M \setminus \text{int}_M M_i) \approx \mathcal{H}(M_i, \text{bd}_M M_i)$$

and the latter is locally contractible by [11, Corollary 7.3], the conclusion follows from Lemmas 6.4 and 4.9 (2). \square

6.2. Homeomorphism groups of non-compact surfaces.

In this subsection, we shall recognize the local topological type of the pair $(\mathcal{H}(M, K), \mathcal{H}_c(M, K))$ for a 2-manifold M and a closed polyhedral subset $K \subset M$. Suppose M is a σ -compact 2-manifold possibly with boundary. Then M admits a combinatorial triangulation unique up to PL-homeomorphisms [26]. We fix a triangulation of M and regard M as a PL 2-manifold. A *subpolyhedron* of M means a subpolyhedron with respect to this PL-structure. A 2-submanifold of M means a subpolyhedron N of M such that N is a 2-manifold and $\text{bd}_M N$ is transverse to ∂M so that $\text{bd}_M N$ is a proper 1-submanifold of M and $M \setminus \text{int}_M N$ is also a 2-manifold. Let $\mathcal{H}^{PL}(M)$ denote the subgroup of $\mathcal{H}(M)$ consisting of PL-homeomorphisms with respect to the PL-structure of M , and set $\mathcal{H}_c^{PL}(M, K) = \mathcal{H}^{PL}(M) \cap \mathcal{H}_c(M, K)$.

Theorem 2 and Proposition 3 in Introduction follow from the next theorem with taking $K = \emptyset$.

Theorem 6.5. *Suppose M is a non-compact σ -compact 2-manifold possibly with boundary and $K \subsetneq M$ is a subpolyhedron.*

- (1) *If $\text{cl}_M(M \setminus K)$ is compact, then (i) $\mathcal{H}(M, K)$ is an l_2 -manifold and hence (ii) $\mathcal{H}_0(M, K)$ is an open normal subgroup of $\mathcal{H}(M, K)$.*
- (2) *If $\text{cl}_M(M \setminus K)$ is non-compact, then*
 - (i) *$(\mathcal{H}(M, K), \mathcal{H}_c(M, K))$ is locally homeomorphic to $(\square^\omega l_2, \square^\omega l_2)$, hence $\mathcal{H}(M, K)$ is locally homeomorphic to $\square^\omega l_2$ and $\mathcal{H}_c(M, K)$ is an $(l_2 \times \mathbb{R}^\infty)$ -manifold,*
 - (ii) *$\mathcal{H}_0(M, K)$ is an open normal subgroup of $\mathcal{H}_c(M, K)$ and thus*

$$\mathcal{H}_c(M, K) \approx \mathcal{H}_0(M, K) \times \mathcal{M}_c(M, K),$$

where $\mathcal{M}_c(M, K) = \mathcal{H}_c(M, K) / \mathcal{H}_0(M, K)$ (with the discrete topology).

- (3) *The subgroup $\mathcal{H}_c^{PL}(M, K)$ is homotopy dense in $\mathcal{H}_c(M, K)$.*

We keep the notations for $G = \mathcal{H}(M)$ listed in Subsection 6.1. For $K \subset L \subset M$, the symbols

$$\mathcal{E}_K(L, M) \supset \mathcal{E}_K^*(L, M) \supset \mathcal{E}_K^*(L, M) = \mathcal{E}_K^G(L, M)$$

denote the space of embeddings and the subspaces of proper embeddings and extendable embeddings, respectively. These spaces are endowed with the compact-open topology.

To prove Theorem 6.5, we use the next two theorems besides Proposition 5.5.

Theorem 6.6. ([15], [23], cf. [32]) *Suppose M is a compact 2-manifold possibly with boundary and $K \subsetneq M$ is a subpolyhedron. Then, (i) $\mathcal{H}(M, K)$ is an l_2 -manifold and (ii) $\mathcal{H}^{PL}(M, K)$ is homotopy dense in $\mathcal{H}(M, K)$.*

The following is a slight extension of the results in [31] and [32]. Note that the assertion was verified in [23] in the most important case that $K = \emptyset$ and L is either a proper arc, an orientation-preserving circle or a compact 2-submanifold of M .

Theorem 6.7. *Suppose M is a 2-manifold possibly with boundary and $K \subset L$ are two subpolyhedra of M such that $\text{cl}_M(L \setminus K)$ is compact.*

- (1) *For every closed subset C of M with $C \cap \text{cl}_M(L \setminus K) = \emptyset$, the restriction map*

$$r : \mathcal{H}(M, K) \rightarrow \mathcal{E}_K^*(L, M), \quad r(h) = h|_L$$

has a local section $s : \mathcal{U} \rightarrow \mathcal{H}_0(M, K \cup C) \subset \mathcal{H}(M, K)$ at the inclusion $i_L : L \subset M$.

- (2) *The restriction map $r : \mathcal{H}(M, K) \rightarrow \mathcal{E}_K^*(L, M)$ is a principal $\mathcal{H}(M, L)$ -bundle.*
 (3) (i) *$\mathcal{E}_K^*(L, M)$ is an open neighborhood of the inclusion i_L in $\mathcal{E}_K^*(L, M)$.*
 (ii) *The spaces $\mathcal{E}_K^*(L, M)$ and $\mathcal{E}_K^*(L, M)$ are l_2 -manifolds if $\dim(L \setminus K) \geq 1$.*

Proof. In [31, Proposition 4.2] and [32, Theorem 2.1], we verified the case where L is compact. The general case is obtained if we choose a regular neighborhood N of $\text{cl}_M(L \setminus K)$ in $M \setminus C$ and apply the compact case to the data

$$(N, (L \cap N) \cup \text{bd}_M N, (K \cap N) \cup \text{bd}_M N). \quad \square$$

Proof of Theorem 6.5. Consider the subgroup $G_K = \mathcal{H}(M, K)$ of $G = \mathcal{H}(M)$ and the collection \mathcal{F} of all compact subpolyhedra of M . Then the transformation group G_K on M has a strong topology with respect to \mathcal{F} . Choose any exhausting sequence $(M_i, L_i, N_i)_{i \in \mathbb{N}}$ for M such that each M_i and N_i are compact 2-submanifolds of M . Recall that $L_i = M_i \setminus \text{int}_M M_{i-1} \subset \text{int}_M N_i$. Then, for each $i \in \mathbb{N}$

- (i) $L_{2i} \in \mathcal{F}$, (ii) (N_{2i}, L_{2i}) has LSP_{G_K} and
 (iii) the quotient topology on the space

$$\mathcal{E}^{G_K}(L_{2i}, M) \cong \mathcal{E}_K^G(K \cup L_{2i}, M) = \mathcal{E}_K^*(K \cup L_{2i}, M)$$

coincides with the compact-open topology.

The assertion (ii) is verified by applying Theorem 6.7(1) to the data $(L, K, C) = (K \cup L_{2i}, K, M \setminus \text{Int } N_{2i})$ and (iii) follows from Remark 4.6(0). Hence, it follows from Proposition 5.5 that

- (*)₁ $(G_K, G_{K,c}) \approx_\ell (\square, \square)_{i \in \mathbb{N}} \mathcal{E}^{G_K}(L_{2i}, M) \times (\square, \square)_{i \in \mathbb{N}} G_K(L_{2i-1})$,
 (*)₂ the multiplication map $p : \square_{i \in \mathbb{N}} G_K(M_i) \rightarrow G_{K,c}$ has a local section at any point of $G_{K,c}$.

(1) Take a compact submanifold N of M such that $\text{cl}_M(M \setminus K) \subset N$. Then $\mathcal{H}(M, K)$ is identified with $\mathcal{H}(N, (N \cap K) \cup \text{bd}_M N)$ and the assertion (1) is the direct consequence of Theorem 6.6.

(2) Since $G_K(L_{2i-1}) = \mathcal{H}(M, K \cup (M \setminus \text{int}_M L_{2i-1})) \approx \mathcal{H}(L_{2i-1}, (K \cap L_{2i-1}) \cup \text{bd}_M L_{2i-1})$ and $L_i \not\subset K$ for infinitely many $i \in \mathbb{N}$, by Theorem 6.6 (i) and Theorem 6.7 (3)(ii) we have

$$(\square, \square)_{i \in \mathbb{N}} \mathcal{E}^{G_K}(L_{2i}, M) \times (\square, \square)_{i \in \mathbb{N}} G_K(L_{2i-1}) \approx_\ell (\square^\omega l_2, \square^\omega l_2).$$

Hence, the assertion (2) follows from $(*)_1$.

(3) Let $H = \mathcal{H}^{PL}(M)$ and consider the subgroup H_K of G_K . Since G_c is paracompact, so is $G_{K,c}$. Since

$$\begin{aligned} (G_K(M_i), H_K(M_i)) &= (\mathcal{H}(M, K \cup K_i), \mathcal{H}^{PL}(M, K \cup K_i)) \\ &\approx (\mathcal{H}(M_i, (M_i \cap K) \cup \text{bd}_M M_i), \mathcal{H}^{PL}(M_i, (M_i \cap K) \cup \text{bd}_M M_i)), \end{aligned}$$

Theorem 6.6 (ii) implies that $H_K(M_i)$ is HD in $G_K(M_i)$. By $(*)_2$, we can apply Lemma 4.9 (3) to assert that $\mathcal{H}_c^{PL}(M, K) = H_{K,c}$ is HD in $\mathcal{H}_c(M, K) = G_{K,c}$. \square

6.3. Diffeomorphism groups of non-compact smooth manifolds.

In this subsection, we study diffeomorphism groups of non-compact smooth manifolds endowed with the Whitney C^∞ -topology. Suppose M is a smooth σ -compact n -manifold without boundary. Let $\mathcal{D}(M)$ denote the group of diffeomorphisms of M endowed with the Whitney C^∞ -topology (= the very-strong C^∞ -topology in [19]).

The topological group $G = \mathcal{D}(M)$ admits the natural action on M . Similarly to the previous subsection, we use the following notations for $K, N \subset M$:

$$\begin{aligned} \mathcal{D}_0(M) &= G_0, \quad \mathcal{D}_c(M) = G_c, \quad \mathcal{D}(M, K) = G_K, \\ \mathcal{D}(M, M \setminus N) &= G(N), \quad \mathcal{D}(M, K \cup (M \setminus N)) = G_K(N), \\ \mathcal{D}_c(M, K) &= G_{K,c}, \quad \mathcal{D}_0(M, K) = (G_K)_0. \end{aligned}$$

Since the inclusion map $\mathcal{D}(M) \subset \mathcal{H}(M)$ is continuous, it follows from Proposition 4.1 that $\mathcal{D}_0(M) \subset \mathcal{D}_c(M)$. The quotient group $\mathcal{M}_c^\infty(M) = \mathcal{D}_c(M)/\mathcal{D}_0(M)$ (with the quotient topology) is called the mapping class group of M .

Let \mathcal{F}_c^∞ denote the collection of all compact smooth n -submanifolds of M . For $L \in \mathcal{F}_c^\infty$ and a subset $K \subset L$, let $\mathcal{E}_K^\infty(L, M)$ denote the space of C^∞ -embeddings $f : L \rightarrow M$ with $f|_K = \text{id}_K$ and let $\mathcal{E}_K^{\infty,*}(L, M) = \mathcal{E}_K^G(L, M)$ (the space of extendable C^∞ -embeddings). These spaces are endowed with the compact-open C^∞ -topology (and its subspace topology). There is a natural restriction map

$$r : \mathcal{D}(M, K) \rightarrow \mathcal{E}_K^\infty(L, M), \quad r(h) = h|_L.$$

The following is the main result of this subsection.

Theorem 6.8. *Suppose M is a non-compact σ -compact smooth n -manifold without boundary.*

- (1) $(\mathcal{D}(M), \mathcal{D}_c(M)) \approx_\ell (\square^\omega l_2, \square^\omega l_2)$. Hence, $\mathcal{D}(M) \approx_\ell \square^\omega l_2$ and $\mathcal{D}_c(M)$ is an $(l_2 \times \mathbb{R}^\infty)$ -manifold.
- (2) $\mathcal{D}_0(M)$ is an open normal subgroup of $\mathcal{D}_c(M)$ and

$$\mathcal{D}_c(M) \approx \mathcal{D}_0(M) \times \mathcal{M}_c^\infty(M).$$

In the proof, we use Proposition 5.5 and the following bundle theorem (cf. [9], [18], [22], [27], [29]).

Theorem 6.9. *Suppose K and L are smooth n -submanifolds of M such that they are closed subsets of M , $K \subset \text{int}_M L$ and $\text{cl}_M(L \setminus K)$ is compact and nonempty.*

- (1) *For any closed subset C of M with $C \cap L = \emptyset$, the restriction map $r : \mathcal{D}(M, K) \rightarrow \mathcal{E}_K^\infty(L, M)$ has a local section*

$$s : \mathcal{U} \rightarrow \mathcal{D}(M, K \cup C) \subset \mathcal{D}(M, K)$$

at the inclusion $i_L : L \subset M$ such that $s(i_L) = \text{id}_M$.

- (2) *The spaces $\mathcal{D}(M, K \cup (M \setminus L))$ and $\mathcal{E}_K^\infty(L, M)$ are infinite-dimensional separable Fréchet manifolds (thus topological l_2 -manifolds) and $\mathcal{E}_K^{\infty,*}(L, M)$ is an open subset of $\mathcal{E}_K^\infty(L, M)$.*

Proof of Theorem 6.8. We follow the formulation of Subsections 5.1–5.2 and apply Proposition 5.5 to the transformation group $G = \mathcal{D}(M)$ on M . For $L \in \mathcal{F}_c^\infty$, the compact-open C^∞ -topology on $\mathcal{E}^G(L, M)$ is an admissible topology, and Theorem 6.9(1) and Remark 4.4(iii) imply that this topology coincides with the quotient topology $\hat{\tau}^G(L, M)$ on $\mathcal{E}^G(L, M)$ induced by the restriction map $r : G \rightarrow \mathcal{E}^G(L, M)$. Hence, it is seen that the transformation group G has a strong topology with respect to \mathcal{F}_c^∞ .

Now we can apply Proposition 5.5 to any exhausting sequence $(M_i, L_i, N_i)_{i \in \mathbb{N}}$ for M such that each M_i is a compact n -submanifold of M . For each $i \in \mathbb{N}$, it is seen that $L_{2i} \in \mathcal{F}_c^\infty$ and (N_{2i}, L_{2i}) has LSP_G by Theorem 6.9(1). Hence, from Proposition 5.5 it follows that

$$(G, G_c) \approx_\ell (\square, \square)_{i \in \mathbb{N}} \mathcal{E}^G(L_{2i}, M) \times (\square, \square)_{i \in \mathbb{N}} G(L_{2i-1}).$$

The latter pair is locally homeomorphic to $(\square^\omega l_2, \square^\omega l_2)$ by Theorem 6.9(2).

Due to Theorem 6.9(2), $G(L)$ is separable metrizable for each compact smooth n -submanifold $L \subset M$. Then G_c is paracompact by Proposition 4.1(2). \square

REFERENCES

- [1] S. Alpern and V. Prasad, *End behaviour and ergodicity for homeomorphisms of manifolds with finitely many ends*, Canad. J. Math. **39** (2) (1987), 473–491.
- [2] R. D. Anderson, *Spaces of homeomorphisms of finite graphs*, unpublished preprint.
- [3] R. D. Anderson and R. H. Bing, *A complete elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines*, Bull. Amer. Math. Soc. **74** (1968), 771–792.
- [4] T. Banakh, *On hyperspaces and homeomorphism groups homeomorphic to products of absorbing sets and \mathbb{R}^∞* , Tsukuba J. Math. **23** (1999), 495–504.
- [5] T. Banakh, K. Mine and K. Sakai, *Classifying homeomorphism groups of infinite graphs*, Topology Appl. **157** (2009), 108–122.
- [6] T. Banakh, K. Mine, K. Sakai and T. Yagasaki, *Topological groups locally homeomorphic to LF -spaces*, preprint.
- [7] T. Banakh and D. Repoš, *A topological characterization of LF -spaces*, preprint (arXiv:0911.0609).
- [8] T. Banakh and T. Yagasaki, *The diffeomorphism groups of the real line are pairwise bihomeomorphic*, in: Proceedings of “Infinite-Dimensional Analysis and Topology”, Yaremche, Ivano-Frankivsk, Ukraine, 2009; a special issue in Topology, in press, online November 2009 (arXiv:0804.3645v3).
- [9] J. Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France **89** (1961), 227–380.
- [10] A. V. Černavskiĭ, *Local contractibility of the group of homeomorphisms of a manifold*, (Russian) Mat. Sb. (N.S.) **79** (121) (1969), 307–356.
- [11] R. D. Edwards and R. C. Kirby, *Deformations of spaces imbeddings*, Ann. of Math. **93** (1971), 63–88.
- [12] R. Engelking, *General Topology*, Revised and completed edition, Sigma Ser. in Pure Math. **6**, Heldermann Verlag, Berlin, 1989.
- [13] D. B. Gauld, *The graph topology for function spaces*, Indian J. Math. **18** (1976), 125–132.
- [14] R. Geoghegan, *On spaces of homeomorphisms, embeddings and functions – I*, Topology **11** (1972), 159–177.
- [15] R. Geoghegan and W. E. Haver, *On the space of piecewise linear homeomorphisms of a manifold*, Proc. Amer. Math. Soc. **55** (1976), 145–151.
- [16] G. Gruenhage, *Generalized metric spaces*, In: K. Kunen and J. Vaughan (eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 423–501.
- [17] I. I. Guran and M. M. Zarichnyi, *The Whitney topology and box products*, Dokl. Akad. Nauk Ukrain. SSR Ser. A. 1984, no.11, 5–7.

- [18] R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) **7**:1 (1982), 65–222.
- [19] S. Illman, *The very-strong C^∞ topology on $C^\infty(M, N)$ and K -equivariant maps*, Osaka J. Math. **40**, no. 2 (2003), 409–428.
- [20] A. Kriegl and P. W. Michor, *The Convenient Setting of Global Analysis*, Math. Surveys and Monog. **53**, Amer. Math. Soc., Providence, R.I., 1997.
- [21] R. C. Kirby and L. C. Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Annals of Math. Studies **88**, Princeton University Press, Princeton, N.J., 1977.
- [22] J. A. Leslie, *On a differential structure for the group of diffeomorphisms*, Topology **6** (1967), 263–271.
- [23] R. Luke and W. K. Mason, *The space of homeomorphisms on a compact two-manifold is an absolute neighborhood retract*, Trans. Amer. Math. Soc. **164** (1972), 275–285.
- [24] P. Mankiewicz, *On topological, Lipschitz, and uniform classification of LF -spaces*, Studia Math. **52** (1974), 109–142.
- [25] E. Michael, \aleph_0 -spaces, J. Math. Mech. **15** (1966), 983–1002.
- [26] E. E. Moise, *Geometric Topology in Dimensions 2 and 3*, GTM **47**, Springer-Verlag, New York-Heidelberg, 1977.
- [27] R. S. Palais, *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. **34** (1960), 305–312.
- [28] F. Quinn, *Ends of maps III, Dimensions 4 and 5*, J. Differential Geom. **17**(3) (1982), 503–521.
- [29] R. T. Seeley, *Extension of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc. **15** (1964), 625–626.
- [30] H. Toruńczyk, *Absolute retracts as factors of normed linear spaces*, Fund. Math. **86** (1974), 53–67.
- [31] T. Yagasaki, *Spaces of embeddings of compact polyhedra into 2-manifolds*, Topology Appl. **108** (2000), 107–122.
- [32] T. Yagasaki, *Homotopy types of homeomorphism groups of noncompact 2-manifolds*, Topology Appl. **108** (2000), 123–136.
- [33] J. E. West, *Open problems in infinite-dimensional topology*, in: J. van Mill and G. M. Reed, (eds.), Open Problems in Topology, Elsevier Sci. Publ. B.V., Amsterdam, 1990, 523–597.
- [34] S. Williams, *Box products*, In: K. Kunen and J. Vaughan (eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 169–200.

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